On synthesis of linear quantum stochastic systems by pure cascading*

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Abstract

Recently, it has been demonstrated that an arbitrary linear quantum stochastic system can be realized as a cascade connection of simple one degree of freedom quantum harmonic oscillators together with a direct interaction Hamiltonian which is bilinear in the canonical operators of the oscillators. However, from an experimental point of view, realizations by pure cascading, without a direct interaction Hamiltonian, would be much simpler to implement and this raises the natural question of what class of linear quantum stochastic systems are realizable by cascading alone. This paper gives a precise characterization of this class of linear quantum stochastic systems and then it is proved that, in the *weaker* sense of transfer function realizability, *all* passive linear quantum stochastic systems belong to this class. A constructive example is given to show the transfer function realization of a two degrees of freedom passive linear quantum stochastic system by pure cascading.

Keywords: Linear quantum stochastic systems, quantum system realization, quantum networks, quantum control, linear quantum optics

1 Background and Motivation

Recently, there has been interest in the literature on control of a linear quantum stochastic system with a controller which is a quantum system of the same type [1, 2, 3, 4], often referred to as "coherent-feedback control". The potential applications for linear quantum stochastic systems include quantum information processing and photonic signal processing. For instance, they can act as the coherent photonic circuitry subsystem in a cavity QED system, the latter system being realized by placing suitable atoms inside the optical cavities

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in a linear quantum stochastic system. Cavity QED networks are of interest for quantum information processing (see, e.g., [5]), such as in the quantum internet [6], whilst the controller realized in [4] is an early sample application of linear quantum stochastic systems to photonic signal processing.

The studies on coherent-feedback control naturally led to the consideration of the network synthesis problem for linear quantum stochastic systems [7], which may be viewed as a quantum analogue of the network synthesis problem for linear electrical systems [8]. Nurdin, James and Doherty [7] have shown that any linear quantum stochastic system can, in principle, be synthesized by a cascade of simple one degree of freedom harmonic oscillators together with a direct interaction Hamiltonian between the canonical operators of these oscillators. Alternative schemes have subsequently been proposed in [9, 10], but we note that [10] considers a weaker type of realizability than in [7, 9], i.e., transfer function realizability (cf. section 3), and the results therein limited to a certain sub-class of linear quantum stochastic systems.

From an experimental perspective, direct bilinear interaction Hamiltonians between independent harmonic oscillators are challenging to implement for systems that have more than just a few degrees of freedom and therefore it becomes important to investigate what kind of systems can be realized by a pure cascade connection. A key result of this paper is a necessary and sufficient condition for a linear quantum stochastic system to be realizable by only a cascade connection of one degree of freedom oscillators, without any direct interaction Hamiltonian. Moreover, we also show that the associated transfer functions of *all* passive linear quantum stochastic systems can always be realized by a cascade connection, proving in general the partial results of [10] without the additional assumptions made therein.

The organization of this paper is as follows. Section 2 sets up the notations and gives a brief overview of linear quantum stochastic systems. Section 3 defines the synthesis problem and discusses the notions of strict realizability and transfer function realizability. Section 4 derives a necessary and sufficient condition for a linear quantum system to be realizable by a pure cascade connection of one degree of freedom quantum harmonic oscillators, in both the strict and transfer function sense of realizability. Section 5 then introduces the class of passive linear quantum systems and proves that all such systems are transfer functions realizable by a pure cascade connection. Finally, section 6 offers some conclusions of this paper.

2 Preliminaries

2.1 Notation

We shall use the following notations: $\mathfrak{i} = \sqrt{-1}$, * denotes the adjoint of a linear operator as well as the conjugate of a complex number. If $A = [a_{jk}]$ then $A^{\#} = [a_{jk}^*]$, and $A^{\dagger} = (A^{\#})^T$, where ^{T} denotes matrix transposition. $\Re\{A\} = (A + A^{\#})/2$ and $\Im\{A\} = \frac{1}{2\mathfrak{i}}(A - A^{\#})$, and denote the identity matrix by I whenever its size can be inferred from context and

use I_n to denote an $n \times n$ identity matrix. Similarly, 0 denotes a matrix with zero entries whose dimensions can be determined from context. $\mathrm{diag}(M_1, M_2, \ldots, M_n)$ denotes a block diagonal matrix with square matrices $M_1, M_2, \ldots, M_n$ on its diagonal block, and $\mathrm{diag}_n(M)$ a block diagonal matrix with the square matrix M appearing on its diagonal blocks n times.

2.2 The class of linear quantum stochastic systems

In this paper, we will be concerned with a class of quantum stochastic models of open (i.e., quantum systems that can interact with an environment) *Markov* quantum systems that are widely used and are standard in quantum optics. Such models have been in the physics and mathematical physics literature since the 1980's, see, e.g., [11, 12, 13, 14, 15]. In particular, we focus on the special sub-class of *linear* quantum stochastic models, see, e.g., [15, section 6.6], [14, sections 3, 3.4.3, 5.3, chapters 7 and 10], [16, section 4], [17, section 5], [18, 2, 3, 7, 19, 4, 20, 21]. These linear quantum stochastic models describe such quantum optical devices as optical cavities [22, section 5.3.6][23, chapter 7], linear quantum amplifiers [14, chapter 7], and finite bandwidth squeezers [14, chapter 10]. Following the terminology in [2, 3, 7], we shall refer to this class of models as *linear quantum stochastic systems*.

Suppose we have n independent quantum harmonic oscillators labelled $1, \ldots, n$. Each oscillator j has position and momentum operators q_j and p_j, respectively. The position and momentum operators satisfy the canonical commutation relations $[q_j, p_k] = 2i\delta_{jk}$, $[q_j, q_k] = 0$, and $[p_j, p_k] = 0$, where δ_{jk} denotes the Kronecker delta that takes on the value 1 only if $j = k$, but is otherwise 0. Equivalently, we may describe them in terms of the $2n$ annihilation and creation operators $a_1, a_1^*, a_2, a_2^*, \ldots, a_n, a_n^*$, with $a_j = (q_j + ip_j)/2$, satisfying the canonical commutation relations $[a_j, a_k^*] = \delta_{jk}$, $[a_j, a_k] = 0$ and $[a_j^*, a_k^*] = 0$. The independent oscillators can be coupled to one or more external independent quantum fields, say m of them. In a Markov quantum system, the m independent fields are essentially *quantum noises* modelled by bosonic annihilation field operators $\mathcal{A}_1(t), \mathcal{A}_2(t), \ldots, \mathcal{A}_m(t)$ that can be defined on a separate Fock space (over $L^2(\mathbb{R})$) for each field operator [11, 13, 24]. For each $\mathcal{A}_j(t)$ there is a corresponding creation field operator $\mathcal{A}_j^*(t)$ that is defined on the same Fock space and is the operator adjoint of $\mathcal{A}_j(t)$, i.e., $\mathcal{A}_j^*(t) = \mathcal{A}_j(t)^*$. The field operators are adapted quantum stochastic processes with forward differentials $d\mathcal{A}_j(t) = \mathcal{A}_j(t + dt) - \mathcal{A}_j(t)$ and $d\mathcal{A}_j^*(t) = \mathcal{A}_j^*(t + dt) - \mathcal{A}_j^*(t)$ that have the quantum Itô products [11, 13, 24]:

$$d\mathcal{A}_j(t)d\mathcal{A}_k(t)^* = \delta_{jk}dt; \ d\mathcal{A}_j^*(t)d\mathcal{A}_k(t) = 0; \ d\mathcal{A}_j(t)d\mathcal{A}_k(t) = 0;$$
$$d\mathcal{A}_j^*(t)d\mathcal{A}_k^*(t) = 0; \ d\mathcal{A}_k(t)dt = 0; \ d\mathcal{A}_k^*(t)dt = 0.$$

More informally, as in the quantum Langevin formalism, we can express $\mathcal{A}_j(t) = \int_0^t \eta_j(s)ds$ and $\mathcal{A}_j^*(t) = \int_0^t \eta_j^*(s)ds$, where $\eta_j(t)$ for $j = 1, \ldots, m$ are independent quantum white noise processes satisfying the informal commutation relations $[\eta_j(s), \eta_k^*(t)] = \delta_{jk}\delta(t - s)$ and $[\eta_j(s), \eta_k(t)] = [\eta_j(s)^*, \eta_k^*(t)] = 0$, where $\eta_j^*(t) = \eta_j(t)^*$, and $\delta(t)$ denotes the Dirac delta function.

Let us collect the position and momentum operators in the column vector x defined as $x = (q_1, p_1, q_2, p_2, \ldots, q_n, p_n)^T$. Note that in terms of x, we may write the canonical commutation relations as $xx^T - (xx^T)^T = 2i\Theta$ with $\Theta = \mathrm{diag}_n(J)$. We take the composite system of n quantum harmonic oscillators to have a *quadratic Hamiltonian* H given by $H = \frac{1}{2}x^T R x$, where R is a real $2n \times 2n$ symmetric matrix. The oscillators are coupled to the quantum field m via the informal singular interaction Hamiltonian $H_m = \mathfrak{i}(L_m \eta_m^*(t) - L_m^* \eta_m(t))$ [12, 14], where $L_m = K_m x$ with $K_m \in \mathbb{C}^{1 \times 2n}$ is a linear coupling operator of the oscillator position and momentum operators to $\eta_m(t)$. Collect the coupling operators $L_1, L_2, \ldots, L_m$ together in one *linear coupling vector* $L = (L_1, L_2, \ldots, L_m)^T = Kx$, with $K = [\ K_1^T \quad K_2^T \quad \ldots \quad K_m^T\ ]^T$, and the field operators together as $\mathcal{A}(t) = (\mathcal{A}_1(t), \mathcal{A}_2(t), \ldots, \mathcal{A}_m(t))^T$. Then the *joint* evolution of the oscillators and the quantum fields is given by a unitary adapted process $U(t)$ satisfying the Hudson-Parthasarathy quantum stochastic differential equation (QSDE) [11, 13, 24, 25]:

$$dU(t) = (\mathrm{tr}((S - I)^T d\Lambda(t)) + d\mathcal{A}(t)^\dagger L - L^\dagger S d\mathcal{A}(t) - (\mathfrak{i}H + \frac{1}{2}L^\dagger L dt))U(t),$$

where $S \in \mathbb{C}^{m \times m}$ is a complex unitary matrix (i.e., $S^\dagger S = SS^\dagger = I$) called the *scattering matrix*, and $\Lambda(t) = [\Lambda_{jk}(t)]_{j,k=1,\ldots,m}$. The processes $\Lambda_{jk}(t)$ for $j, k = 1, \ldots, m$ are adapted quantum stochastic processes that are referred to as *gauge processes*, and the forward differentials $d\Lambda_{jk}(t) = \Lambda_{jk}(t + dt) - \Lambda_{jk}(t)$ $j, k = 1, \ldots, m$ have the quantum Itô products:

$$d\Lambda_{jk}(t)d\Lambda_{j'k'}(t) = \delta_{kj'}d\Lambda_{jk'}(t), \quad d\mathcal{A}_j(t)d\Lambda_{kl}(t) = \delta_{jk}d\mathcal{A}_l(t), \quad d\Lambda_{jk}d\mathcal{A}_l(t)^* = \delta_{kl}d\mathcal{A}_j^*(t),$$

with all other remaining cross products between $d\Lambda_{jk}(t)$ and either of dt, $d\mathcal{A}_{j'}(t)$ or $d\mathcal{A}_{k'}^*(t)$ being zero. Informally, we may express $\Lambda_{jk}(t) = \int_0^t \eta_j^*(s)\eta_k(s)ds$.

For any adapted processes $V(t)$ and $W(t)$ satisfying a quantum Ito stochastic differential equation, we have the *quantum Ito rule* $d(V(t)W(t)) = V(t)dW(t) + (dV(t))W(t) + dV(t)dW(t)$. Using the quantum Ito rule and the quantum Ito products given above, as well as exploiting the canonical commutation relations between the operators in x, the *Heisenberg evolution* $X(t) = U(t)^* x U(t)$ of the canonical operators in the vector x satisfies the quantum stochastic differential equation, see [16, section 4], [17, section 5], [2, 7]:

$$dX(t) = d(U(t)^* x U(t)) = \tilde{A}X(t)dt + \tilde{B}\left[\begin{array}{c} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{array}\right]; X(0) = x,$$

$$dY(t) = d(U(t)^* \mathcal{A}(t) U(t)) = \tilde{C}x(t)dt + \tilde{D}d\mathcal{A}(t), \tag{1}$$

with $\tilde{A} = 2\Theta(R + \Im\{K^\dagger K\})$, $\tilde{B} = 2i\Theta[\ -K^\dagger S \quad K^T S^\#\ ]$, $\tilde{C} = K$, and $\tilde{D} = S$, where $Y(t) = (Y_1(t), \ldots, Y_m(t))^T = U(t)^* \mathcal{A}(t) U(t)$ is a vector of *output fields* that results from the interaction of the quantum harmonic oscillators and the incoming quantum fields $\mathcal{A}(t)$. Note that the dynamics of $X(t)$ is linear, while $Y(t)$ depends linearly on $X(t)$ and $\mathcal{A}(t)$. We refer to n as the *degrees of freedom* of the oscillators. If $n = 1$, we shall often refer to the linear quantum stochastic system as a one degree of freedom (open quantum harmonic) oscillator.

Following [25], we denote a linear quantum stochastic system with Hamiltonian H, coupling vector L and scattering matrix S simply as $G = (S, L, H)$ or $G = (S, Kx, \frac{1}{2}x^T Rx)$.

We also recall the *concatenation product* $\boxplus$ and *series product* $\triangleleft$ for open Markov quantum systems [25] defined by $G_1 \boxplus G_2 = (\mathrm{diag}(S_1, S_2), (L_1^T, L_2^T)^T, H_1 + H_2)$, and $G_2 \triangleleft G_1 = (S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \Im\{L_2^\dagger S_2 L_1\})$. Since both products are associative, the products $G_1 \boxplus G_2 \boxplus \ldots \boxplus G_n$ and $G_n \triangleleft G_{n-1} \triangleleft \ldots \triangleleft G_1$ are unambiguously defined.

3   Synthesis of linear quantum stochastic systems

The network synthesis problem for linear quantum stochastic systems can be stated (in a strict sense, as explained below) as the problem of how to systematically realize a given linear quantum stochastic system with a given fixed set of matrix parameters S, K, R from a bin of certain basic quantum optical components; see [7] for details of these basic components. A particular solution was proposed to the synthesis problem, see [7, Theorem 5.1]: Any linear quantum stochastic system with n degrees of freedom can be synthesized via a quantum network consisting of a cascade connection of n one degree of freedom harmonic oscillators together with a direct interaction Hamiltonian that is bilinear in the canonical operators of the oscillators. Partition R as $R = [R_{jk}]_{j,k=1,\ldots,n}$ with $R_{jk} \in \mathbb{R}^{2\times 2}$ and K as $K = [\, K_1 \quad K_2 \quad \ldots \quad K_n \,]$ with $K_k \in \mathbb{C}^{m\times 2}$. Then according to [7, Theorem 5.1] a system $G = (S, Kx, \frac{1}{2}x^T R x)$ can be decomposed as $G = (G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1) \boxplus (0, 0, H^d)$, where the G_i's are (simpler) one degree of freedom open harmonic oscillators $G_i = (S_i, K_i x_i, \frac{1}{2}x_i^T R_{ii} x_i)$ $(x_i = (q_i, p_i)^T)$ with parameter values specified by the theorem, and H^d is a direct bilinear interaction Hamiltonian of the form $H^d = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} x_j^T \left(R_{jk} - \Im\{K_k^\dagger K_j\}^T \right) x_k$. The work [7] then shows how each of the G_i's can be synthesized from the bin of given components and how H^d can be realized. However, in current practical experiments, implementation of H^d can be challenging for systems that have more than just a few degrees of freedom. Therefore, it is of interest to characterize the class of systems that can be synthesized by pure cascade connection alone, that is, with $H^d \equiv 0$.

As alluded to at the beginning of this section, we emphasize that [7] considers a *strict* type of realization problem, that is, it deals with how to synthesize a *given* and *fixed* triplet $\{S, L = Kx, H = \frac{1}{2}x^T R x\}$ that describes a linear quantum stochastic system G. This type of strict realizability is relevant, for instance, in cases where the internal dynamics $X(t)$ may represent some (continuous time) quantum information processing algorithm and thus needs to be realized as given. However, for some linear quantum control problems such as robust disturbance attenuation [2] and LQG synthesis [3], the internal dynamics are inconsequential. In this case there is freedom to modify/transform these dynamics and what is important is the associated (classical) complex transfer function associated with the system matrices (A, B, C, D)[1]. As is well known, a transfer function is invariant under a

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[1]As in [2], here we shall not define the transfer function of quantum systems, but associate to a quantum system G with system matrices (A, B, C, D) a *classical, doubled-up* [26], transfer function $G(s) = [\, C^T \quad C^\dagger \,]^T (sI - A)^{-1} B + \mathrm{diag}(D, D^\#)$. However, we also remark that $G(s)$ can actually be properly interpreted as a genuine transfer function for the quantum system following [26, 21, 27, 1], this being a common practice in the physics community via Fourier transform methods [23, 14]. In any case, we are dealing with the same object $G(s)$ and thus the particular interpretation attached to it becomes immaterial for our purpose.

similarity transformation of the system matrices $(A, B, C, D) \mapsto (VAV^{-1}, VB, CV^{-1}, D)$ for any invertible matrix V. However, for linear quantum systems the transformation matrix V for a similarity transformation is restricted in that it has to be a *symplectic* matrix: V is real and satisfies the condition $V\Theta V^T = \Theta$. This ensures that the transformed variable $Z(t) = VX(t)$ satisfies the required canonical commutation relations (CCR) of quantum mechanics: $Z(t)Z(t)^T - (Z(t)Z(t)^T)^T = 2i\Theta$, so the system remains physical. Note that the set of all symplectic matrices of a fixed dimension form a group and in particular V^{-1} is again a symplectic matrix. Such a similarity transformation in quantum systems corresponds to replacing $G = (S, Kx, \frac{1}{2}x^T Rx)$ with $G' = (S, KV^{-1}x, \frac{1}{2}x^T V^{-T} RV^{-1}x)$. This motivates us to introduce the following definition:

Definition 1 *Let $G = (S, Kx, \frac{1}{2}x^T Rx)$ and $G' = (S', K'x, \frac{1}{2}x^T R'x)$ be two linear quantum stochastic systems. Then G' is said to be* transfer function equivalent *to G or is a* transfer function realization *of G if $S' = S$ and there exists a symplectic matrix V such that $R' = V^{-T} RV^{-1}$, $K' = KV^{-1}$ (or, equivalently, $R = V^T R'V$ and $K = K'V$). G is then said to be* transfer function realizable *by G', and vice-versa.*

Remark 2 *It is important to note that two transfer function equivalent systems G and G' will not necessarily generate the same input-output dynamics $(A(t), Y(t))$ for all $t \geq 0$. This is because although they can have different parameters, they always have the same initial value $X(0) = x$, whilst for quantum systems x clearly cannot be zero due to the CCR condition. If the A matrix of G is Hurwitz then the input-output dynamics of G and G' converge in the limit $t \to \infty$. However, as remarked earlier, for some linear quantum control design objectives internal dynamics and initial conditions do not play an essential role, only the transfer function does.*

4 Conditions for realizability by a pure cascade connection

In this section we state and prove a theorem that characterizes the class of linear quantum stochastic systems that can be realized simply by a cascade connection of one degree of freedom (open quantum harmonic) oscillators. Let us first introduce the following notation: $S_{k \leftarrow j} = S_k \cdots S_{j+1} S_j$ for all $j < k$, $S_{k \leftarrow k} = S_k$ and $S_{k \leftarrow k+1} = I_m$, and let $x_i = (q_i, p_i)^T$ for $i = 1, \ldots, n$ so that $x = (x_1^T, \ldots, x_n^T)^T$, where $xx^T - (xx^T)^T = 2i\Theta$. Moreover, we introduce the following terminology: A square matrix F is said to be lower 2×2 block triangular if it has a lower block triangular form when partitioned into 2×2 blocks:

$$
F = \begin{bmatrix}
F_{11} & 0_{2\times2} & 0_{2\times2} & \ldots & 0_{2\times2} \\
F_{21} & F_{22} & 0_{2\times2} & \ldots & 0_{2\times2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
F_{n1} & F_{n2} & \ldots & \ldots & F_{nn}
\end{bmatrix},
$$

where F_{jk}, $j \leq k$, is of dimension 2×2. We start with the following lemma:

Lemma 3 *The cascade connection $G_n \lhd G_{n-1} \lhd \cdots \lhd G_1$ of one degree of freedom harmonic oscillators $G_i = (S_i, K_i x_i, \frac{1}{2} x_i^T R_i x_i)$ $(i = 1, \ldots, n)$ realizes a linear quantum stochastic system $G = (S, Kx, \frac{1}{2} x^T R x)$ with $S = S_{n\leftarrow1}$, $K = \begin{bmatrix} S_{n\leftarrow2} K_1 & S_{n\leftarrow3} K_2 & \ldots & K_n \end{bmatrix}$, $R = [R_{ij}]_{i,j=1,\ldots,n}$, where $R_{jj} = R_j$, $R_{kj} = \Im\{K_k^\dagger S_{k\leftarrow j+1} K_j\}$ whenever $k < j$ and $R_{jk} = R_{kj}^T$ whenever $j > k$. In particular, $R + \Im\{K^\dagger K\}$ is lower 2×2 block triangular.*

Proof. The proof proceeds along the lines of the proof of [7, Theorem 5.1]. By the series product formula (cf. section 2.2) for the cascade of two one degree of freedom oscillators $G_1 = (S_1, K_1 x_1, \frac{1}{2} x_1^T R_1 x_1)$ and $G_2 = (S_2, K_2 x_2, \frac{1}{2} x_2^T R_2 x_2)$, we get the oscillator $G_{(2)} = G_2 \lhd G_1 = (S_2 S_1, S_2 K_1 x_1 + K_2 x_2, \frac{1}{2} x_1^T R_1 x_1 + \frac{1}{2} x_2^T R_2 x_2 + x_2^T \Im\{K_2^\dagger S_2 K_1\} x_1)$. Letting $x_{(2)} = (x_1^T, x_2^T)^T$, the latter may be compactly written as: $G_{(2)} = (S_{(2)}, K_{(2)} x_{(2)}, \frac{1}{2} x_{(2)}^T R_{(2)} x_{(2)}^T)$ with $S_{(2)} = S_{2\leftarrow1} = S_2 S_1$, $K_{(2)} = \begin{bmatrix} S_2 K_1 & K_2 \end{bmatrix}$ and $R_{(2)} = \begin{bmatrix} R_1 & \Im\{K_2^\dagger S_{2\leftarrow2} K_1\}^T \\ \Im\{K_2^\dagger S_{2\leftarrow2} K_1\} & R_2 \end{bmatrix}$.

Repeating the computation for $G_{(k)} = G_k \lhd G_{(k-1)}$ iteratively for $k = 3, \ldots, n-1$ and writing $x_{(k)} = (x_1^T, x_2^T, \ldots, x_k^T)^T$ and $G_{(k)} = (S_{(k)}, K_{(k)} x_{(k)}, \frac{1}{2} x_{(k)}^T R_{(k)} x_{(k)})$ at each iteration k, we arrive at the desired result with $G = G_{(n)}$, $S = S_{(n)}$, $K = K_{(n)}$ and $R = R_{(n)}$ as stated in the lemma.

To see that $R + \Im\{K^\dagger K\}$ is lower 2×2 block triangular, we note that $K^\dagger K$ may be expressed as follows:

$$
K^\dagger K = \begin{bmatrix}
K_1^\dagger K_1 & K_1^\dagger S_2^\dagger K_2 & K_1^\dagger S_{3\leftarrow2}^\dagger K_2 & \ldots & K_1^\dagger S_{n\leftarrow2}^\dagger K_n \\
K_2^\dagger S_2 K_1 & K_2^\dagger K_2 & K_2^\dagger S_3^\dagger K_3 & \ldots & K_2^\dagger S_{n\leftarrow3}^\dagger K_n \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
K_n^\dagger S_{n\leftarrow2} K_1 & K_n^\dagger S_{n\leftarrow3} K_2 & \ldots & K_n^\dagger S_n K_{n-1} & K_n^\dagger K_n
\end{bmatrix}.
$$

Note that since $K^\dagger K$ is by definition a Hermitian matrix, the 2×2 block elements above the diagonal blocks are the Hermitian transpose of the corresponding elements below the diagonal blocks. It follows therefore that the imaginary part of the block $(K^\dagger K)_{jk}$ at block row j and block column k must satisfy the relation: $\Im\{(K^\dagger K)_{jk}\} = -\Im\{(K^\dagger K)_{kj}\}^T$. However, from the expression for R derived above and its symmetry, we already have that if $k > j$:

$$
R_{jk} = R_{kj}^T = \Im\{K_k^\dagger S_{k\leftarrow j+1} K_j\}^T = \Im\{(K^\dagger K)_{kj}\}^T.
$$

Therefore, the off-diagonal upper block elements of R cancel those of $\Im\{K^\dagger K\}$ when they are summed and we conclude that the matrix $R + \Im\{K^\dagger K\}$ is a lower 2×2 block triangular matrix.

Recall again the partitioning of R as $R = [R_{jk}]_{j,k=1,\ldots,n}$ with $R_{jk} \in \mathbb{R}^{2 \times 2}$ and of K as $K = \begin{bmatrix} K_1 & K_2 & \ldots & K_n \end{bmatrix}$ with $K_k \in \mathbb{C}^{m \times 2}$. We may now state the following result:

Theorem 4 *A linear quantum stochastic system $G = (S, Kx, \frac{1}{2} x^T R x)$ with n degrees of freedom is realizable by a pure cascade of n one degree of freedom harmonic oscillators (without a direct interaction Hamiltonian) if and only if the A matrix given by $A = 2\Theta(R + \Im\{K^\dagger K\})$ is a lower block triangular matrix with blocks of size 2×2. If this condition is*

satisfied then G can be explicitly constructed as the cascade connection $G_n \lhd G_{n-1} \lhd \ldots \lhd G_1$ with $G_1 = (S, K_1 x_1, \frac{1}{2} x_1^T R_{11} x_1)$, and $G_k = (I, K_k x_k, \frac{1}{2} x_k^T R_{kk} x_k)$ for $k = 2, \ldots, n$.

Proof. The proof of the only if part follows directly from Lemma 3, as follows. If G can be realized by a pure cascade connection of n one degree of freedom harmonic oscillators then by the lemma, $R + \Im\{K^\dagger K\}$ is a lower 2×2 block triangular matrix. However, since Θ is 2×2 block diagonal, it follows that the matrix $A = 2\Theta(R + \Im\{K^\dagger K\})$ is also a lower 2×2 block triangular matrix.

Conversely, the if part of the proof can be shown by explicitly constructing a pure cascade connection of n one degree oscillators that realizes G. If the A matrix associated with G is lower 2×2 block triangular then so is the matrix $\frac{1}{2}\Theta^{-1}A = -\frac{1}{2}\Theta A = R + \Im\{K^\dagger K\}$. As we already saw in the proof of Lemma 3, this structure implies that $R_{jk} = \Im\{K_j^\dagger K_k\}$ whenever $k > j$ and $R_{kj} = \Im\{K_j^\dagger K_k\}^T$ if $k < j$. Now, using the notation of Lemma 3, let us define the one degree of freedom harmonic oscillators G_k for $k = 1, \ldots, n$ as $G_1 = (S, K_1 x_1, \frac{1}{2} x_1^T R_{11} x_1)$, and $G_k = (I, K_k x_k, \frac{1}{2} x_k^T R_{kk} x_k)$ for $k = 2, \ldots, n$. It follows from Lemma 3 that $G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 = (S, Kx, \frac{1}{2} x^T R x)$. That is, this cascade connection realizes G.

Theorem 4 has a direct consequence on the weaker notion of transfer function realization of a linear quantum system. The main result is the following corollary:

Corollary 5 *A linear quantum system $G = (S, Kx, \frac{1}{2} x^T R x)$ is transfer function realizable by a pure cascade connection of one degree of freedom harmonic oscillators if and only if there is a symplectic transformation matrix V such that the linear quantum stochastic system $G' = (S, KV^{-1}x, \frac{1}{2} x^T V^{-T} R V^{-1} x)$ has an A matrix which is lower 2×2 block triangular.*

Proof. By Definition 1, G is transfer function realizable by a pure cascade connection if and only if there exists a symplectic matrix V such that $G' = (S, KV^{-1}x, \frac{1}{2} x^T V^{-T} R V^{-1} x)$ is realizable by a pure cascade connection. But from Theorem 4 this is true if and only if the A matrix associated with G' (i.e., $A = 2\Theta(R' + \Im\{K'^\dagger K'\})$) is lower 2×2 block triangular.

5   Passive linear quantum stochastic systems

In this section it will be shown that the class of passive linear quantum stochastic systems (as defined below) are transfer function realizable by a cascade connection. In [10] it has been shown by a constructive algorithm that a "generic" sub-class of such systems are transfer function realizable by pure cascading, the generic systems being required to satisfy assumptions on the distinctness of the eigenvalues and invertibility of certain matrices. In this section we remove such assumptions, and show by exploiting the algebraic structure of passive systems that the result is valid in general for *all* passive systems.

For $k = 1, \ldots, n$, let $a_k = (q_k + \mathfrak{i}p_k)/2$ be the annihilation operators for mode k and define $a = (a_1, \ldots, a_n)^T$. Then a satisfies the CCR $\begin{bmatrix} a \\ a^\# \end{bmatrix} [\ a^\dagger \quad a^T\ ] - \left(\begin{bmatrix} a^\# \\ a \end{bmatrix} [\ a^T \quad a^\dagger\ ] \right)^T = \mathrm{diag}(I_n, -I_n)$. Moreover, note that $(a^T, a^\dagger)^T = [\ \Sigma^T \quad \Sigma^\dagger\ ]^T x$ with

$$\Sigma = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\mathfrak{i} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\mathfrak{i} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{2} & \frac{1}{2}\mathfrak{i} \end{bmatrix}.$$

We also make note that $\begin{bmatrix} \Sigma \\ \Sigma^\# \end{bmatrix}^{-1} = 2\,[\ \Sigma^\dagger \quad \Sigma^T\ ]$ and from the relation $\begin{bmatrix} \Sigma \\ \Sigma^\# \end{bmatrix} 2\,[\ \Sigma^\dagger \quad \Sigma^T\ ] = I$ we have the identities:

$$\Sigma\Sigma^\dagger = I/2 = \Sigma^\#\Sigma^T; \ \Sigma\Sigma^T = 0 = \Sigma^\#\Sigma^\dagger. \tag{2}$$

Therefore, we also have

$$x = \begin{bmatrix} \Sigma \\ \Sigma^\# \end{bmatrix}^{-1} \begin{bmatrix} a \\ a^\# \end{bmatrix} = 2\,[\ \Sigma^\dagger \quad \Sigma^T\ ] \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

A system $G = (S, Kx, \frac{1}{2}x^T Rx)$ is said to be *passive* if we can write $H = \frac{1}{2}x^T Rx = \frac{1}{2}a^\dagger \tilde{R}a + c$ and $L = Kx = \tilde{K}a$ for some complex $n \times n$ *Hermitian* matrix $\tilde{R}$, a complex $m \times n$ (here m again denotes the number of input and output fields in and out of G) matrix $\tilde{K}$, and some *real* constant c. As discussed in [9], here the term passive for such systems is physically motivated since they can be implemented using only passive components like optical cavities, mirrors, beamsplitters and phase shifters; this follows from Theorem 5.1 of [7] and the constructions shown in section 6 of that paper. Also shown in [9], we can express $\frac{1}{2}a^\dagger \tilde{R}a$ and $\tilde{K}a$ in the form $\frac{1}{2}a^\dagger \tilde{R}a = \frac{1}{2}x^T \Re\{\Sigma^\dagger \tilde{R}\Sigma\}x - \frac{1}{4}\sum_{j=1}^n \tilde{R}_{jj}$ and $\tilde{K}a = \tilde{K}\Sigma x$. Therefore, we may set $R = \Re\{\Sigma^\dagger \tilde{R}\Sigma\}$ and $K = \tilde{K}\Sigma$. Note also from [9] that the 2×2 block diagonal elements $\{R_{jj}; \ j = 1, \ldots, n\}$ is of the form $R_{jj} = \lambda_j I_2$ for some $\lambda_j \in \mathbb{R}$ for all j. Now we shall derive some properties of A and show that there exists a unitary and symplectic matrix that transforms it into a lower 2×2 block triangular matrix.

Lemma 6 $[\ \Sigma^T \quad \Sigma^\dagger\ ]^T A\,[\ \Sigma^\dagger \quad \Sigma^T\ ] = \mathrm{diag}(M, M^\#)$, *where* $M = \frac{1}{2}\Sigma\Theta\Sigma^\dagger(\tilde{R} - \mathfrak{i}\tilde{K}^\dagger \tilde{K})$.

Proof. For the proof, we exploit the identities (2) as well as the following easily verified

identities: $\Sigma\Theta\Sigma^T = 0 = \Sigma^{\#}\Theta\Sigma^{\dagger}$. Using these identities, we have the following:

$$
\begin{aligned}
[\ \Sigma^T \quad \Sigma^{\dagger}\ ]^T A\,[\ \Sigma^{\dagger} \quad \Sigma^T\ ] \ &=\ 2[\ \Sigma^T \quad \Sigma^{\dagger}\ ]^T \Theta(R + \Im\{K^{\dagger}K\})\,[\ \Sigma^{\dagger} \quad \Sigma^T\ ] \\
&=\ 2[\ \Sigma^T \quad \Sigma^{\dagger}\ ]^T \Theta(\Re\{\Sigma^{\dagger}\tilde{R}\Sigma\} + \Im\{\Sigma^{\dagger}\tilde{K}^{\dagger}\tilde{K}\Sigma\})\,[\ \Sigma^{\dagger} \quad \Sigma^T\ ] \\
&=\ 2[\ \Sigma^T \quad \Sigma^{\dagger}\ ]^T \Theta\Big(\frac{1}{2}(\Sigma^{\dagger}\tilde{R}\Sigma + \Sigma^T\tilde{R}^{\#}\Sigma^{\#}) \\
&\qquad -\frac{\mathrm{i}}{2}(\Sigma^{\dagger}\tilde{K}^{\dagger}\tilde{K}\Sigma - \Sigma^T\tilde{K}^T\tilde{K}^{\#}\Sigma^{\#})\Big)\,[\ \Sigma^{\dagger} \quad \Sigma^T\ ] \\
&=\ 2[\ \Sigma^T \quad \Sigma^{\dagger}\ ]^T \Theta\,[\ \tfrac{1}{4}\Sigma^{\dagger}\tilde{R} - \tfrac{\mathrm{i}}{4}\Sigma^{\dagger}\tilde{K}^{\dagger}\tilde{K} \quad \tfrac{1}{4}\Sigma^T\tilde{R}^{\#} + \tfrac{\mathrm{i}}{4}\Sigma^T\tilde{K}^T\tilde{K}^{\#}\ ] \\
&=\ \mathrm{diag}\Big(\frac{1}{2}\Sigma\Theta\Sigma^{\dagger}\tilde{R} - \frac{\mathrm{i}}{2}\Sigma\Theta\Sigma^{\dagger}\tilde{K}^{\dagger}\tilde{K}, \\
&\qquad \frac{1}{2}\Sigma^{\#}\Theta\Sigma^T\tilde{R}^{\#} + \frac{\mathrm{i}}{2}\Sigma^{\#}\Theta\Sigma^T\tilde{K}^T\tilde{K}^{\#}\Big).
\end{aligned}
$$

Then we have the following theorem:

Theorem 7 *Let U be the complex unitary matrix in a* Schur decomposition *of the matrix M of Lemma 6: $M = U^{\dagger}\hat{M}U$, where $\hat{M}$ is a lower triangular matrix. Then the matrix*

$$
V = 2\,[\ \Sigma^{\dagger} \quad \Sigma^T\ ]\,\mathrm{diag}(U, U^{\#})[\ \Sigma^T \quad \Sigma^{\dagger}\ ]^T
$$

is a real*, unitary, and* symplectic *matrix that transforms A into a lower 2×2 block triangular matrix: $V A V^{\dagger} = \hat{A}$, where $\hat{A}$ is a real lower 2×2 block triangular matrix. Therefore, every passive linear quantum system has a transfer function realization by pure cascading and such a realization is obtained by applying the construction of Theorem 4 to $G' = (S, KV^T x, \frac{1}{2}x^T V R V^T x)$. Moreover, each of the one degree of freedom oscillator in the cascade will also be passive.*

Proof. The existence of U is guaranteed by the well known result that every complex matrix M has a Schur decomposition of the form $M = U^{\dagger}\hat{M}U$ with $\hat{M}$ lower triangular. Note then that we also have $\hat{M}^{\#} = U^{\#}M^{\#}U^T$. Let V be as defined in the theorem. Then by Lemma 6 the following is true:

$$
V A V^{\dagger} = 4\,[\ \Sigma^{\dagger} \quad \Sigma^T\ ]\,\mathrm{diag}(\hat{M}, \hat{M}^{\#})[\ \Sigma^T \quad \Sigma^{\dagger}\ ]^T = 4(\Sigma^{\dagger}\hat{M}\Sigma + \Sigma^T\hat{M}^{\#}\Sigma^{\#}) = 8\Re\{\Sigma^{\dagger}\hat{M}\Sigma\}.
$$

Now, since $\hat{M}$ is a lower triangular matrix, it follows by inspection (using the special structure of Σ) that $\hat{A} = 8\Re\{\Sigma^{\dagger}\hat{M}\Sigma\}$ is a lower 2×2 block diagonal matrix, as claimed. That V is real follows from the fact that we may write $V = 2(\Sigma^{\dagger}U\Sigma + \Sigma^T U^{\#}\Sigma^{\#}) = 4\Re\{\Sigma^{\dagger}U\Sigma\}$. That it is *unitary* follows from the observation that $\sqrt{2}[\ \Sigma^{\dagger} \quad \Sigma^T\ ]$ and $\sqrt{2}\begin{bmatrix} \Sigma \\ \Sigma^{\#} \end{bmatrix}$ are unitary (as a consequence of (2)) and that $\mathrm{diag}(U, U^{\#})$ is also unitary. To see that V is also *symplectic* define $b = Ua$ and $z = Vx$. By the unitarity of U we have that b and $b^{\#}$ satisfy

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the same the commutation relations as a and $a^{\#}$ (i.e., b is again an annihilation operator). Then we have

$$z = Vx = V2[\; \Sigma^\dagger \;\; \Sigma^T \;]\begin{bmatrix} a \\ a^{\#} \end{bmatrix} = 2[\; \Sigma^\dagger \;\; \Sigma^T \;]\mathrm{diag}(U, U^{\#})\begin{bmatrix} a \\ a^{\#} \end{bmatrix} = [\; \Sigma^\dagger \;\; \Sigma^T \;]\begin{bmatrix} b \\ b^{\#} \end{bmatrix}.$$

Now, this implies that z consists of the canonical position and momentum operators associated with the modes in b and satisfies the same CCR as x. But since $z = Vx$ and V is real, preservation of the CCR implies that V is necessarily a symplectic matrix (this is standard knowledge in quantum mechanics; see, e.g., [28, section III]).

Using the fact that $V^{-1} = V^T = V^\dagger$ established above, it follows from Theorem 4 that the passive quantum system G is transfer function equivalent to $G' = (S, KV^T x, \frac{1}{2} x^T V R V^T x)$ whose A matrix is lower 2×2 block triangular. Let $K' = KV^T = [\; K_1' \;\; K_2' \;\; \ldots \;\; K_n' \;]$ and $R' = VRV^T = [R'_{jk}]_{j,k=1,\ldots,n}$. By Theorem 4 we have $G' = G_n \lhd G_{n-1} \lhd \ldots \lhd G_1$ with $G_k = (S_k, K_k' x_k, \frac{1}{2} x_k^T R'_{kk} x_k)$, $S_1 = S$ and $S_k = I$ for $k > 1$. We now show that each G_k is passive. Recall that $K = \tilde{K}\Sigma$ and write $\tilde{K} = [\; \tilde{K}_1 \;\; \ldots \;\; \tilde{K}_n \;]$ with $\tilde{K}_k \in \mathbb{C}^{m \times 1}$. Using (2) we have that $K'x = KV^\dagger x = \tilde{K}U^\dagger a$. By expanding both sides of the equality $K'x = \tilde{K}U^\dagger a$ and collecting and equating terms of the same index, it follows that $K_k' x_k = (\tilde{K}U^\dagger)_k a_k$, where $(\tilde{K}U^\dagger)_k$ is the k-th $\mathbb{C}^{m \times 1}$ block component of $\tilde{K}U^\dagger$. On the other hand, since G is passive we have that $R_{kk} = \lambda_k I_2$ for some $\lambda_k \in \mathbb{R}$, and recalling that $R = \Re\{\Sigma^\dagger \tilde{R}\Sigma\}$, it follows by inspection (after some algebraic manipulations using (2)) that $R'_{kk} = \lambda_k' I_2$ for some $\lambda_k' \in \mathbb{R}$. Thus, $x_k^T R'_{kk} x_k = \lambda_k' a_k^* a_k + c$ for some constant c and we conclude that each G_k is also passive.

Example 8 *Let $G = (I, \tilde{K}a, \frac{1}{2}a^\dagger \tilde{R}a + \frac{5}{4})$ be a passive system with $\tilde{R} = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$ and $\tilde{K} = \begin{bmatrix} 1+0.5i & -2+i \\ -5-2i & 3-4i \end{bmatrix}$. Here $K = \begin{bmatrix} 0.5+0.25i & -0.25+0.5i & -1+0.5i & -0.5-i \\ -2.5-i & 1-2.5i & 1.5-2i & 2+1.5i \end{bmatrix}$ and $R = \begin{bmatrix} 0.5I_2 & 0.25(I_2 - J) \\ 0.25(I_2 + J) & 0.75I_2 \end{bmatrix}$. By Lemma 6 and Theorem 7, we have that $U = \begin{bmatrix} -0.6933+0.0039i & 0.2244-0.6849i \\ 0.7204+0.0209i & 0.2312-0.6536i \end{bmatrix}$ and $\hat{M} = \begin{bmatrix} -14.8390-0.7912i & 0 \\ 0.6344-0.2225i & -0.2235-0.4588i \end{bmatrix}$. Then by the formula of Theorem 7 we have*

$$V = \begin{bmatrix} -0.6933 & 0.0039 & 0.2244 & 0.6849 \\ -0.0039 & -0.6933 & -0.6849 & 0.2244 \\ 0.7204 & -0.0209 & 0.2312 & 0.6536 \\ 0.0209 & 0.7204 & -0.6536 & 0.2312 \end{bmatrix}.$$

Let $K' = KV^{-1} = [\; K_1' \;\; K_2' \;]$, then

$$K' = \begin{bmatrix} -0.9144-0.7441i & 0.7441-0.9144i & -0.1926-0.3684i & 0.3684-0.1926i \\ 3.4433+1.2621i & -1.2621+3.4433i & -0.1679-0.1501i & 0.1501-0.1679i \end{bmatrix},$$

and

$$R' = VRV^{-1} = \begin{bmatrix} R'_{11} & R'_{12} \\ (R'_{12})^T & R'_{22} \end{bmatrix} = \begin{bmatrix} 0.7912 & 0 & 0.1113 & 0.3172 \\ 0 & 0.7912 & -0.3172 & 0.1113 \\ 0.1113 & -0.3172 & 0.4588 & 0 \\ 0.3172 & 0.1113 & 0 & 0.4588 \end{bmatrix}.$$

Therefore, by the theorem, G is transfer function realizable by $G' = (I, K'x, \frac{1}{2}x^T R'x)$. It is easily checked that $R' + \Im\{K'K\}$ is lower 2×2 block triangular:

$$R' + \Im\{K'^{\dagger}K'\} = \begin{bmatrix} 0.7912 & 14.8390 & 0 & 0 \\ -14.8390 & 0.7912 & 0 & 0 \\ 0.2225 & -0.6344 & 0.4588 & 0.2235 \\ 0.6344 & -0.2225 & -0.2235 & 0.4588 \end{bmatrix},$$

therefore G' can be realized by a pure cascade connection of one degree of freedom harmonic oscillators. According to Theorem 4, $G' = G_2 \triangleleft G_1$ with $G_1 = (I, K'_1x_1, \frac{1}{2}x_1^T R'_{11}x_1)$ and $G_2 = (I, K'_2x_2, \frac{1}{2}x_2^T R'_{22}x_2)$. It is easily inspected that both G_1 and G_2 are passive. A quantum optical realization of G' is illustrated in Fig. 1.
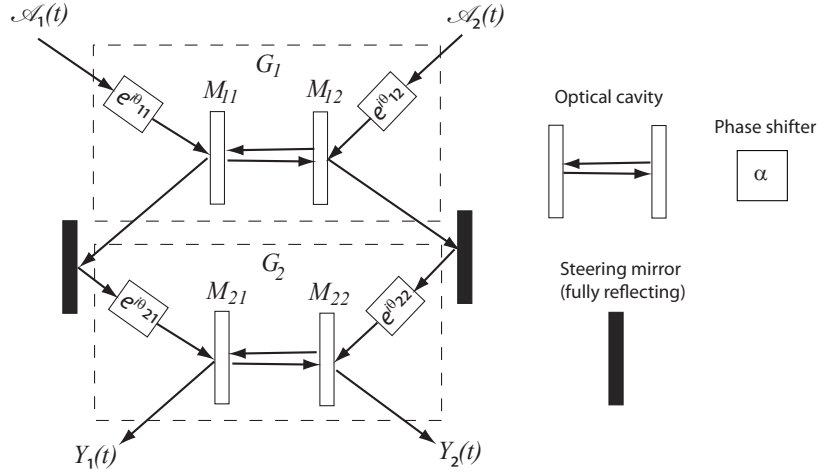

Figure 1: Realization of G' as the cascade connection of G_1 and G_2. G_1 and G_2 are each realized by an optical cavity and a phase shifter; see, e.g., [22, 23, 7] for a discussion of these devices. Dark rectangles depict fully reflecting mirrors, while light rectangles depict partially transmitting mirrors; an optical cavity is formed by bouncing light back and forth between two mirrors. Here $\theta_{11} = -2.4585$, $\theta_{12} = 0.3513$, $\theta_{21} = -2.0525$, and $\theta_{22} = -2.4121$, and the partially transmitting mirror M_{jk} in the optical cavity G_j, $j, k = 1, 2$, have the coupling coefficients $\gamma_{11} = 1.3898$, $\gamma_{12} = 13.4492$, $\gamma_{21} = 0.1728$ and $\gamma_{22} = 0.0507$, respectively. The resonance frequencies of the optical cavities of G_1 and G_2 have a detuning of $0.7912 - 0.4588 = 0.3324$, with G_1 having the higher resonance frequency.

6   Conclusions

We have derived a characterization of linear quantum stochastic systems that can be realized, in a strict or transfer function sense, by a cascade connection of one degree of freedom quantum oscillators alone, without requiring any direct bilinear interaction Hamiltonian between these oscillators. The results are constructive in that if a system can be realized by a cascade connection, it is explicitly shown how to construct it. Then it was shown that the sub-class of passive linear quantum stochastic systems is always transfer function realizable by a pure cascade connection.

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